The Solow residual with non-renewable resources

Production function is

$$Y = AF(K, L, S) \tag{8}$$

where S is the stock of the resource, $F(\cdot, \cdot, \cdot)$ is homogeneous of degree 1 and where A is an index of (Hicks-neutral) productivity.

Take the derivative of (8) with respect to time t:

$$\dot{Y} = AF_K\dot{K} + AF_L\dot{L} + AF_S\dot{S} + F(K, L, S)\dot{A} \tag{9}$$

Divide (9) by Y and do some judicious multiplication by 1 (K/K, L/L, S/S and A/A):

$$\frac{\dot{Y}}{Y} = \frac{K \cdot AF_K}{Y}\frac{\dot{K}}{K} + \frac{L \cdot AF_L}{Y}\frac{\dot{L}}{L} + \frac{S \cdot AF_L}{Y}\frac{\dot{S}}{S} + \cancel{\frac{AF(K, L, S)}{Y}}\frac{\dot{A}}{A} \tag{10}$$

Note that $AF(K, L, S) = Y = K \cdot AF_K + L \cdot AF_L + S \cdot AF_L$ from Euler's theorem for homogeneous functions. Divide this expression through by Y:

$$\frac{Y}{Y} = 1 = \underbrace{\frac{K \cdot AF_K}{Y}}_{\alpha_K} + \underbrace{\frac{L \cdot AF_L}{Y}}_{\alpha_L} + \underbrace{\frac{S \cdot AF_S}{Y}}_{\alpha_S} \tag{11}$$

Substitute the definitions for α_K, α_L and α_S in (10) as well as those for growth rates ($g_Y \equiv \dot{Y}/Y$, etc):

$$g_Y = \alpha_K g_K + \alpha_L g_L + \alpha_S g_F + g_A \tag{12}$$

Substitute $\alpha_K = 1 - \alpha_L - \alpha_S$ from (11):

$$\begin{aligned} g_Y &= (1 - \alpha_L - \alpha_S)g_K + \alpha_L g_L + \alpha_S g_S + g_A \\ &= g_K + \alpha_L(g_L - g_K) + \alpha_S(g_S - g_K) + g_A \end{aligned} \tag{13}$$

Re-arrange (13):

$$\underbrace{g_Y - g_K - \alpha_L(g_L - g_K)}_{\text{Solow residual}} = g_A + \alpha_S(g_S - g_K) \tag{14}$$