The Solow residual

Production function is

$$Y = AF(K, L) \tag{1}$$

where $F(\cdot, \cdot)$ is homogeneous of degree 1 (i.e., constant returns to scale) and where A is an index of (Hicks-neutral) productivity.

Take the derivative of (1) with respect to time t:

$$\dot{Y} = AF_K\dot{K} + AF_L\dot{L} + F(K, L)\dot{A} \tag{2}$$

where dots indicate the derivative with respect to time ($\dot{Y} \equiv \partial Y/\partial t$, $\dot{K} \equiv \partial K/\partial t$, etc) and where $F_K \equiv \partial F(K, L)/\partial K$ and $F_L \equiv \partial F(K, L)/\partial L$ are the marginal products of capital and labour, respectively.

Divide (2) by Y and do some judicious multiplication by 1 (K/K, L/L and A/A):

$$\frac{\dot{Y}}{Y} = \frac{K \cdot AF_K}{Y}\frac{\dot{K}}{K} + \frac{L \cdot AF_L}{Y}\frac{\dot{L}}{L} + \cancel{\frac{AF(K, L)}{Y}}\frac{\dot{A}}{A} \tag{3}$$

Note that $AF(K, L)/Y = 1$ from (1).

Euler's theorem for homogeneous functions gives us $AF(K, L) = Y = K \cdot AF_K + L \cdot AF_L$. Divide this expression through by Y:

$$\frac{Y}{Y} = 1 = \underbrace{\frac{K \cdot AF_K}{Y}}_{\alpha_K} + \underbrace{\frac{L \cdot AF_L}{Y}}_{\alpha_L} \tag{4}$$

Substitute the definitions for α_K and α_L in (3) and denote the growth rates $\dot{Y}/Y$, $\dot{K}/K$, $\dot{L}/L$ and $\dot{A}/A$ by g_Y, g_K, g_L and g_A, respectively:

$$g_Y = \alpha_K g_K + \alpha_L g_L + g_A \tag{5}$$

Substitute $\alpha_K = 1 - \alpha_L$ from (4):

$$\begin{aligned} g_Y &= (1 - \alpha_L)g_K + \alpha_L g_L + g_A \\ &= g_K + \alpha_L(g_L - g_K) + g_A \end{aligned} \tag{6}$$

Re-arrange (6) to obtain the 'Solow residual'

$$\underbrace{g_Y - g_K - \alpha_L(g_L - g_K)}_{\text{Solow residual}} = g_A \tag{7}$$